$$q(\tau) = -\frac{1}{3}\frac{dG(\tau)}{d\tau}. \quad (9b)$$

Once the function $G(\tau)$ is determined from the solution of equations (8), the hemispherical reflectivity R and the transmissivity T of the slab are determined according to the foregoing definitions from the following relations.

$$R = \frac{1}{2\pi}\left[G(0) + \frac{2}{3}\frac{dG(0)}{d\tau}\right] \quad (10)$$

$$T = -\frac{1}{3\pi}\frac{dG(\tau_0)}{d\tau}. \quad (11)$$

RESULTS

Tables 1 and 2 show respectively the hemispherical reflectivity and the transmissivity of the slab obtained from the exact analysis and the P_1-approximation for several different values of the optical thickness, single scattering albedo and the boundary surface reflectivities. The absorptivity of the slab can also be determined from the data presented in these tables since the sum of the absorptivity, reflectivity and transmissivity is equal to unity. The exact analysis shows that the reflectivity of the slab is slightly higher with specularly reflecting boundary at $\tau = \tau_0$ than with diffusely reflecting boundary. For optical thicknesses 15 and larger the hemispherical reflectivity is almost equal to that of a semi-infinite medium and transmissivity becomes almost zero. The results with the P_1-approximation, however, do not distinguish whether the reflectivity at the boundary surface $\tau = \tau_0$ is specular or diffuse. The P_1-approximation underestimates the hemispherical reflectivity, and the accuracy of this approximation is not so good for smaller values of ω; for some cases $\omega < 0{\cdot}2$ it has shown negative results which are meaningless. However, for ω close to unity and large optical thicknesses the P_1-approximation gives reasonably good results.

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MONTE CARLO RADIATION SOLUTIONS—EFFECT OF ENERGY PARTITIONING AND NUMBER OF RAYS

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INTRODUCTION

IN RECENT years, Monte Carlo methods have found wide acceptance as a tool for solving radiation heat transfer problems, as documented, for example, in Howell's excellent survey paper [1]. The present paper describes a generalized Monte Carlo technique in which the energy content of a ray bundle is partitioned into two portions. One portion is governed by deterministic laws whereas the other portion is treated probabilistically, that is, by Monte Carlo methods. One of the advantages of the energy partitioning approach is brought into focus when the heat transfer results are monitored continuously as the successive ray bundles are released from the emitting surface. Whereas the results from a conventional Monte Carlo solution may fluctuate substantially as a function of the number of rays, those from the energy partitioning approach are much more stable. In addition, acceptable numerical results appear to be obtainable with less computer time when the partitioning approach is used.

To illustrate the method, consideration is given here to the apparent emittance of isothermal-walled cavities, with specific numerical solutions for diffusely emitting and reflecting conical cavities. In general, the method is applicable to any situation where a portion of the energy content of a ray bundle is governed by deterministic laws.

THE ENERGY PARTITIONING METHOD

Consider an isothermal-walled cavity (temperature T) having wall area A_w and wall graybody emittance ϵ, with an aperture of area A_a. If E_{out} represents the radiant energy streaming out of the aperture,† then the apparent emittance ϵ_a is defined as

$$\epsilon_a = E_{out}/\sigma T^4 A_a \tag{1}$$

where the denominator is the radiant energy that would stream from the cavity if a black surface at temperature T were stretched across the aperture opening. Evidently, the determination of ϵ_a requires that E_{out} be found.

In the Monte Carlo formulation, ray (or photon) bundles are envisioned as being released at various positions along the cavity wall. Each ray bundle is assigned an energy content E^* given by

$$E^* = \epsilon\sigma T^4 A_w/N \tag{2}$$

where N is the number of bundles released. To provide perspective for the energy partitioning method, it is useful first to describe a conventional Monte Carlo treatment of the ray bundles.

When a wall location is established at which a ray bundle is to be released, the direction of departure is determined by drawing random numbers which fix the angles θ and ϕ of a locally implanted spherical coordinate system. Then, the trajectory of the ray is traced and its point of impingement on the cavity wall or on the plane of the aperture is computed. If the ray passes through the aperture, it is tallied, and attention is directed to the next emitted ray. On the other hand, if the ray strikes the cavity wall, a random number is drawn to establish whether it will be absorbed or reflected. If the ray is absorbed, no tally is made and consideration is given to the next ray; whereas if the ray is reflected, it continues its life cycle until it either exits from the cavity or is absorbed within the cavity.

If N_{out} represents the number of rays that leave the cavity, then, taking account of equation (2), it follows that $E_{out} = (N_{out}/N)\epsilon\sigma T^4 A_w$ and from equation (1)

$$\epsilon_a = \epsilon(N_{out}/N)(A_w/A_a). \tag{3}$$

Note that only rays that leave the cavity are tallied. In many cases, N_{out} may be a small fraction of N, so that N may have to be very large in order that N_{out} is large enough to constitute a statistically meaningful sample size. Also note that a ray trace, which is the most time-consuming part of the computation procedure, is required even if a ray is absorbed or exits from the cavity.

In contrast to the foregoing, in the partitioning approach each one of the ray bundles contributes to E_{out} and, in addition, the amount of ray tracing is decreased. The motivation for the partitioning approach stems from the realization that E_{out} can be regarded as having two components. One component is the radiant energy which, subsequent to emission, streams from the cavity opening without intervening reflections at the cavity walls. The magnitude of this component can be calculated from geometrical considerations alone. The second component of E_{out} is the radiant energy which streams from the cavity after experiencing one or more reflections at the cavity walls.

It appears reasonable to incorporate the aforementioned idea into the Monte Carlo formulation. Suppose that the wall location has been established at which the ith ray bundle is to be released. For that location, the fraction F_i of the emitted radiation which passes directly out of the aperture is known (i.e. F_i is an angle factor). Then, the energy content E^* of the ray bundle is partitioned into two portions, F_iE^* and $(1 - F_i)E^*$. Of these, F_iE^* passes directly out of the aperture and is tallied. The other portion, $(1 - F_i)E^*$, remains within the cavity; but before tracing its trajectory, a random number is selected to determine whether or not it will be absorbed at its point of impingement on the cavity walls. If absorption occurs, the ray trace is not performed and attention is directed to a new ray bundle. On the other hand, if the ray is not absorbed, its point of impingement is determined.

At the point of impingement, a second partitioning occurs. Let F_{i1} denote the angle factor of the aperture as

† As is customary, radiant energy that may have originated in the environment outside the aperture is not included in E_{out}.

seen from the point of impingement. Then, the two partitioned portions are $(1 - F_i)F_{i1}E^*$ and $(1 - F_i)(1 - F_{i1})E^*$, the first of which leaves the cavity and is tallied, whereas the second portion remains in the cavity and continues its life cycle until absorption occurs.

From the foregoing, it is seen that each ray bundle contributes at least one tally to the determination of E_{out}. By employing equations (1) and (2), the apparent emittance can be evaluated from the energy partitioning approach as

$$\epsilon_a = \frac{\epsilon(A_w/A_a)}{N}\left[\sum_{i=1}^{N} F_i + \sum_{i=1}^{N} G_i\right] \quad (4)$$

where $G_i = 0$, no reflections
$G_i = (1 - F_i)F_{i1}$, one reflection
$G_i = (1 - F_i)F_{i1} + (1 - F_i)(1 - F_{i1})F_{i2}$, two reflections
and so forth. It should also be noted that a ray trace is performed only when a ray bundle is reflected.

The details of the partitioning method will be further illustrated by application to a conical cavity, which is pictured schematically in the lower portion of Fig. 1. The sketch shows the coordinates and other dimensional nomenclature.

As a prelude to the Monte Carlo computations, the angle factors $F(z)$ were evaluated at a large number of surface locations $0 \leq z \leq L$ and then fitted to high accuracy with least-squares polynomials. For a typical surface location P at which the aperture subtends a solid angle Ω_a, one has

$$F(z) = \int_{\Omega_a} \cos\theta \mathrm{d}\Omega \Big/ \int_{2\pi} \cos\theta \mathrm{d}\Omega. \quad (5)$$

With $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$, equation (5) becomes

$$F(z) = (2/\pi) \int_0^{\pi/2} \phi_1 \cos\theta \sin\theta \, \mathrm{d}\theta \quad (6)$$

where $-\phi_1(\theta, z)$ and $\phi_1(\theta, z)$ are the limits between which ϕ must lie in order that a ray may pass out through the aperture. The angles ϕ_1 were found from the solution of a quadratic equation derived by writing the equation of a ray originating at the point P and intersecting the rim of the aperture. With $\phi_1(\theta, z)$ thus determined, $F(z)$ was evaluated numerically from (6).

The surface location z at which a ray bundle is to be

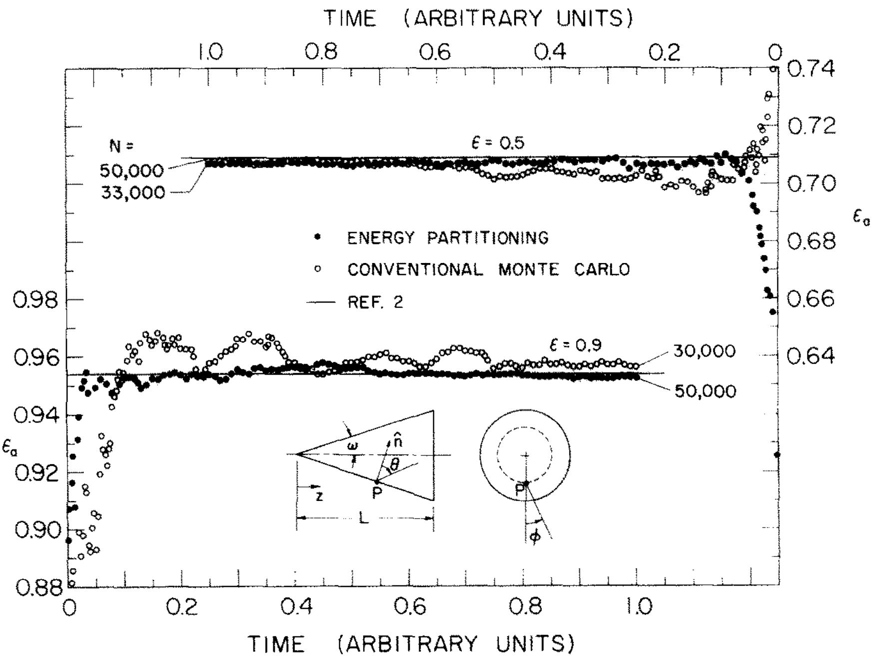

FIG. 1. Apparent emittance results for a conical cavity with half apex angle of 20 degrees.

released and the departure angles θ and ϕ were calculated from the expressions

$$z/L = \sqrt{R_z}, \quad \sin\theta = \sqrt{R_\theta}, \quad \phi = \phi_1 + 2R_\phi(\pi - \phi_1) \qquad (7)$$

where R_z, R_θ and R_ϕ represent random numbers. Absorption of a ray bundle was determined by the condition $R_\alpha \leqslant \alpha$, where α is the graybody absorptance of the cavity wall.

Further details of the computations are available from the authors.

RESULTS AND DISCUSSION

Results for the apparent emittance were obtained both with the energy partitioning approach and the conventional Monte Carlo method. The computations were carried out for cones having half apex angles ω of 20° and 5° and for surface emittances ϵ of 0·9 and 0·5. Preliminary calculations were performed on a CDC 6600 computer to determine computational speeds using the internal system clock. The final runs were made on an SDS 9300 computer.

The ϵ_a results corresponding to $\omega = 20°$ are presented in Fig. 1, with those for $\epsilon = 0{\cdot}9$ referred to the left-hand ordinate and lower abscissa, and those for $\epsilon = 0{\cdot}5$ referred to the right-hand ordinate and upper abscissa. Closed and open circles are used respectively to designate the results from the energy partitioning and conventional Monte Carlo methods. The lower abscissa represents, in arbitrary units, the computer execution times for the runs at $\epsilon = 0{\cdot}9$, whereas the upper abscissa represents, in other arbitrary units, the computer times for the $\epsilon = 0{\cdot}5$ runs. Increasing values of the abscissa correspond to increasing numbers of ray bundles. The total number of ray bundles employed for each case is also indicated in the figure.

Inspection of the figure indicates that ϵ_a may fluctuate substantially as the number of ray bundles increases. This finding appears not to have been previously documented. It suggests the inadvisability of taking the output corresponding to a specific pre-selected number of ray bundles as the answer to the problem. A much safer course is to print out a continuous succession of results and examine them in detail as, for example, in Fig. 1.

The comparison between the results from the partitioning method and the conventional method is highly favorable to the former. The fluctuations exhibited by the partitioning results are very much less, and this is especially evident for $\epsilon = 0{\cdot}9$. Furthermore, to within five in the third significant figure, the partitioning results converge much faster. Indeed, for both $\epsilon = 0{\cdot}9$ and 0·5, convergence to this level of accuracy is attained at 0·1 on the time scales.

The solid lines in the figure represent the results of Lin

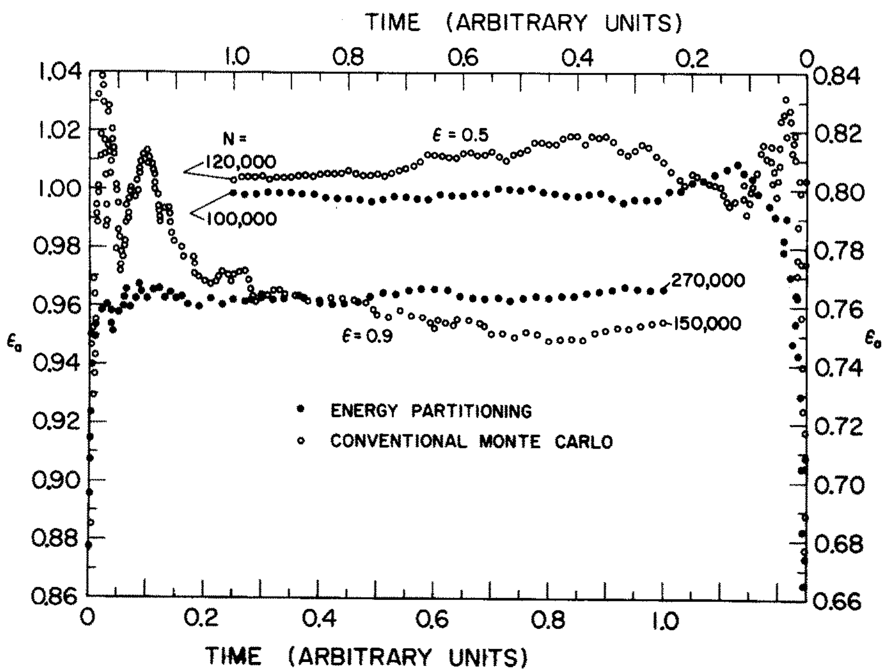

FIG. 2. Apparent emittance results for a conical cavity with half apex angle of 5 degrees.

[2] obtained from numerical solutions of integral equations. Agreement between the present results and those of Lin is seen to be good.

A presentation of ϵ_a results for a cone half angle of 5° is made in Fig. 2. The structure of this figure is similar to that of Fig. 1, and the findings are generally similar. As before, the fluctuations exhibited by the partitioning results are appreciably less than are those in the results from the conventional solution method. Also, the partitioning results converge more rapidly.

The findings presented above indicate that the use of the energy partitioning approach may be highly advantageous in problems where a portion of the energy content of a ray bundle is governed by deterministic laws (e.g. geometrical angle factors). Another interesting outcome of the present work is the documentation of the fluctuations experienced by the Monte Carlo results as the number of rays is increased. This suggests that the output corresponding to a specific pre-selected number of ray bundles may not always be a proper representation of the results.

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HEAT TRANSFER ACROSS TURBULENT FALLING FILMS

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NOMENCLATURE

g,	gravitational acceleration;
h_c,	heat transfer coefficient;
k,	thermal conductivity;
Pr,	Prandtl number;
Re,	Reynolds number $= 4\Gamma/\mu$;
Sc,	Schmidt number;
u^+,	dimensionless velocity $= u/\sqrt{(g\delta)}$;
y^+,	dimensionless coordinate measured from wall $= \dfrac{y\sqrt{(g\delta)}}{\nu}$;
Γ,	flow rate per unit width;
δ,	film thickness;
δ^+,	dimensionless film thickness;
ε,	eddy diffusivity;
ε^+,	$= 1 + \varepsilon/\nu$;
μ,	dynamic viscosity;
ν,	kinematic viscosity;
ρ,	density;
σ,	surface tension.

Subscripts

D,	mass species diffusion;
M,	momentum;
i,	intersection of equations (1) and (4);
t,	turbulent.

RECENTLY Chun and Seban [1] presented the results of an experimental study into heat transfer across evaporating turbulent falling films. They found $h_c \propto Re^{0.4}$, whereas usual analyses e.g. [2,3] based on the conventional hypotheses about turbulent transport that are consistent with pipe flow, predict $h_c \propto Re^{0.2}$ in the limit of high Re (see Fig. 1). Our purpose here is to present an analysis which successfully predicts the Chun–Seban data, and which may also be used for film condensation provided that vapor drag is negligible.

Our starting point is the observation that conventional hypotheses about turbulent transport do apply close to the solid wall, as demonstrated by the electrolytic mass transfer experiments of Iribarne *et al.* [4]. Any one of a number of